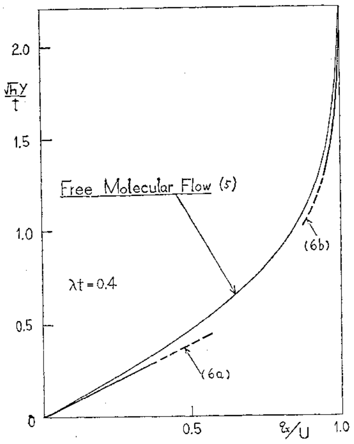

Fig. 1 The distribution of the velocity in the gas ($\lambda t = 0.4$).

and higher order terms. (Note that $h\lambda y^2/t$ may not necessarily be small.) The detailed analysis is similar as in Ref. 2. The preceding result may be compared with the result in Rayleigh flow[1,2] where a solid boundary plays a fundamental role, and a correction proportional to $t^{-1/2}$ to the classical result appears representing a kind of boundary layer with thickness of the order of mean free path. Roughly speaking, for $\lambda t \gg 1$, in the region of viscous diffusion $[y \lesssim (2t/h\lambda)^{1/2}]$, which is much smaller than that of (free) molecular diffusion $(y \lesssim h^{1/2}t)$, mixing and collision of molecules occur adequately to assure the validity of the Navier-Stokes equation except near a solid boundary that does not exist in the present problem. It may also be seen from (6) and (7)** that the effective speed of diffusion is slowed down from $O(h^{-1/2})$ to $O\{(2h\lambda t)^{-1/2}\}$ as time progresses.

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** Note that the solution (6) depends on $h^{1/2}y/t$ except for a small perturbation and that (7) is a function of $\{h^{1/2}\lambda y/(2\lambda t)^{1/2}\}$.

Stability Analysis of a Simplified Flexible Vehicle via Lyapunov's Direct Method

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RECENTLY, it was shown that sufficient conditions for asymptotic stability of equilibrium of certain classes of aeroelastic systems with distributed aerodynamic load can be derived via Lyapunov's direct method.[1] This approach has the advantage over the conventional ones in the respect that it allows one to deal directly with the systems' partial differential equations without resorting to any approximations. Moreover, it is potentially applicable to the stability analysis of nonlinear aeroelastic systems. In this note, similar conditions for a simplified aerodynamic vehicle with flexible tail will be derived using the same approach. This aeroelastic system differs from those considered previously in the sense that the aerodynamic load can be approximated by a concentrated force.

Figure 1 shows the tail portion of a flexible vehicle. For the present analysis, it is assumed that the tail motion corresponds approximately to that of a nonuniform cantilever beam in plane bending. For this system, the dimensionless equation of the perturbed motion about its equilibrium state can be given by

$$m(x)v_0^2l^2 \frac{\partial^2 w(t, x)}{\partial t^2} + v_0 l^3 k_d(t, x) \frac{\partial w(t, x)}{\partial t} = - \frac{\partial^2}{\partial x^2} EI(x) \frac{\partial^2 w(t, x)}{\partial x^2} \quad (1)$$

where both the beam deflection w and the spatial coordinate x have been normalized with respect to the beam length l; t is the dimensionless time normalized with respect to the quantity l/v_0, where v_0 is the freestream velocity of air; m, k_d, and EI are linear mass density, distributed damping coefficient, and bending rigidity, respectively.

Assuming that the aerodynamic load on the tail lifting surface can be approximated by that of a thin flat plate in an incompressible flow,[3] and the incremental aerodynamic moment due to tail motion is negligible, the boundary conditions have the form

$$w(t, 0) = 0 \qquad [\partial w(t, x)/\partial x]|_{x=0} = 0 \quad (2)$$

$$EI(x) \frac{\partial^2 w(t, x)}{\partial x^2}\bigg|_{x=1} = 0 \qquad \frac{\partial}{\partial x} EI(x) \frac{\partial^2 w(t, x)}{\partial x^2}\bigg|_{x=1} = 2\pi\rho_a v_0^2 l^2 ab \left[\frac{\partial w(t, x)}{\partial t} + \frac{\partial w(t, x)}{\partial x}\right]\bigg|_{x=1} \quad (3)$$

where ρ_a is the mass density of the undisturbed air, and a and b are the length and width of the tail lifting surfaces, respectively. The state S_t of this system at any time t can be specified by the functions $w(t, x)$ and $\partial w(t, x)/\partial t$ defined for all $x \epsilon [0, 1]$.

The problem is to derive sufficient conditions for the asymptotic stability of equilibrium of the flexible tail in the sense of Lyapunov[1,2] with respect to a norm defined by

$$||S_t|| = \left\{\int_0^1 \left[\left(\frac{\partial w}{\partial t}\right)^2 + \sum_{n=0}^{2} \left(\frac{\partial^n w}{\partial x^n}\right)^2\right] dx\right\}^{1/2} \quad (4)\dagger$$

Note that although the system is linear, the associated boundary-value problem is nonselfadjoint. The determination of conditions for asymptotic stability in terms of the system parameters is not a trivial task.

To apply Lyapunov's direct method to this problem, consider the following functional:

$$V = \frac{1}{2} \int_0^1 \left[m(x)v_0^2l^2 \left(\frac{\partial w}{\partial t}\right)^2 + 2c_0v_0l^2m(x) \frac{\partial w}{\partial t}\frac{\partial w}{\partial x} + 2\pi\rho_a v_0^2 l^2 ab \left(\frac{\partial w}{\partial x}\right)^2 + EI(x)\left(\frac{\partial^2 w}{\partial x^2}\right)^2\right] dx \quad (5)$$

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† In essence, this norm establishes a measure of the closeness of the system state S_t from the equilibrium null state in terms of the deflection, slope, curvature, and velocity of the beam. Note that for systems having infinite degrees-of-freedom, stability with respect to one norm generally does not imply stability with respect to another norm. The choice of the norm should be based on a careful scrutiny of the physical properties of the particular system under consideration. A detailed discussion of the physical meaning of stability in the sense of Lyapunov is given in Ref. 1.

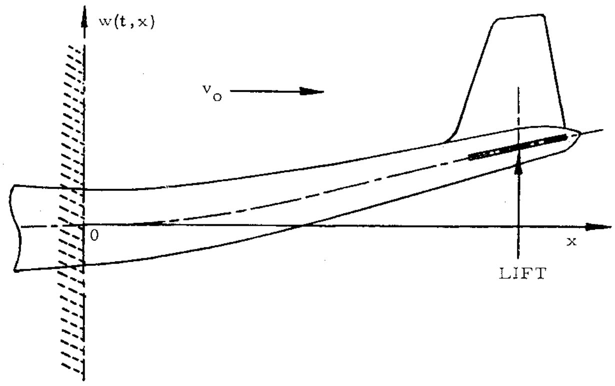

Fig. 1 Tail portion of a flexible vehicle.

where c_0 is a positive constant.

A sufficient condition for asymptotic stability of equilibrium is that V is positive definite with respect to norm (4) and $dV/dt < 0$ along any perturbed motion.[1]

The positive definiteness of V can be readily established by using the following inequality:

$$\int_0^1 G(x)\left(\frac{\partial^2 w(t,x)}{\partial x^2}\right)^2 dx \geq \left[\min_{x\epsilon[0,1]} G(x)\right] \times \int_0^1 \left(\frac{\partial w(t,x)}{\partial x}\right)^2 dx \geq \left[\min_{x\epsilon[0,1]} G(x)\right]\int_0^1 w^2(t,x)dx \quad (6)$$

$$P = \begin{bmatrix} v_0 l^3 k_d(t,x) + \frac{1}{2}c_0 v_0^2 l^2 [dm(x)/dx] & c_0 v_0 l^3 k_d(t,x)/2 & \pi\rho_a v_0^2 l^2 ab \\ c_0 v_0 l^3 k_d(t,x)/2 & \frac{1}{2}c_0(1-\alpha)\min_x |dEI(x)/dx| & 0 \\ \pi\rho_a v_0^2 l^2 ab & 0 & \frac{1}{2}c_0\alpha \min_x |dEI(x)/dx| \end{bmatrix}$$

where G is a positive function of x, and choosing a constant c_0 which satisfies

$$[2\pi\rho_a ab/m(x)]^{1/2} > c_0 > 0 \text{ for all } x\epsilon[0,1] \quad (7)$$

The derivative of V with respect to t, after performing a series of partial integrations, can be expressed in the form

$$\frac{dV}{dt} = \int_0^1 \left(m(x)v_0^2 l^2 \frac{\partial^2 w}{\partial t^2} + \frac{\partial^2}{\partial x^2} EI(x)\frac{\partial^2 w}{\partial x^2}\right)\frac{\partial w}{\partial t}dx - \int_0^1\left(\frac{1}{2}c_0 v_0^2 l^2 \frac{dm(x)}{dx}\left(\frac{\partial w}{\partial t}\right)^2 - c_0 v_0^2 l^2 m(x)\frac{\partial^2 w}{\partial t^2}\frac{\partial w}{\partial x} + 2\pi\rho_a v_0^2 l^2 ab \frac{\partial^2 w}{\partial x^2}\frac{\partial w}{\partial t}\right)dx + \left[2\pi\rho_a v_0^2 l^2 ab\frac{\partial w}{\partial t}\frac{\partial w}{\partial x} + \frac{1}{2}c_0 v_0^2 l^2 m(x)\left(\frac{\partial w}{\partial t}\right)^2 + \left(EI(x)\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial x \partial t} - \frac{\partial}{\partial x}EI(x)\frac{\partial^2 w}{\partial x^2}\right)\frac{\partial w}{\partial t}\right]_0^1 \quad (8)$$

where the differentiability of m has been assumed.

The preceding equation, in view of system equation (1) and boundary conditions (2) and (3), reduces to

$$\frac{dV}{dt} = -\int_0^1\left\{\left[v_0 l^3 k_d(t,x) + \frac{1}{2}c_0 v_0^2 l^2\frac{dm(x)}{dx}\right]\left(\frac{\partial w}{\partial t}\right)^2 + 2\pi\rho_a v_0^2 l^2 ab\frac{\partial^2 w}{\partial x^2}\frac{\partial w}{\partial t}\right\}dx + \int_0^1 c_0 v_0^2 l^2 m(x)\frac{\partial^2 w}{\partial t^2}\frac{\partial w}{\partial x}dx + \left(\frac{1}{2}c_0 v_0^2 l^2 m(1) - 2\pi\rho_a v_0^2 l^2 ab\right)\left(\left.\frac{\partial w}{\partial t}\right|_{x=1}\right)^2 \quad (9)$$

The second integral in (9) can be rewritten in the following form by using (1–3) and integrating by parts:

$$\int_0^1 c_0 v_0^2 l\, m(x)\frac{\partial^2 w}{\partial t^2}\frac{\partial w}{\partial x}dx = -\int_0^1\left[c_0 v_0 l^3 k_d(t,x)\frac{\partial w}{\partial t}\frac{\partial w}{\partial x} - \frac{1}{2}c_0\frac{dEI(x)}{dx}\left(\frac{\partial^2 w}{\partial x^2}\right)^2\right]dx - \frac{1}{2}c_0 EI(0)\left(\left.\frac{\partial^2 w}{\partial x^2}\right|_{x=0}\right)^2 - 2\pi c_0\rho_a v_0^2 l^2 ab\left[\left(\frac{\partial w}{\partial x}\right)^2 + \frac{\partial w}{\partial x}\frac{\partial w}{\partial t}\right]_{x=1} \quad (10)$$

Assuming that

$$[dEI(x)/dx] < 0 \text{ for all } x\epsilon[0,1] \quad (11)$$

the following upper bound for dV/dt can be obtained by substituting (10) into (9) and applying inequality (6):

$$\frac{dV}{dt} \leq -\int_0^1 U'(t,x)PU(t,x)dx - Z'(t)QZ(t) - \frac{1}{2}c_0 EI(0)\left(\left.\frac{\partial^2 w}{\partial x^2}\right|_{x=0}\right)^2 \quad (12)$$

where $(\ )'$ denotes transpose, U and Z are column vectors defined by

$$U = \text{col}\left[\frac{\partial w}{\partial t}, \frac{\partial w}{\partial x}, \frac{\partial^2 w}{\partial x^2}\right] \qquad Z = \text{col}\left[\left.\frac{\partial w}{\partial x}\right|_{x=1} \quad \left.\frac{\partial w}{\partial t}\right|_{x=1}\right]$$

and

$$Q = 2\pi c_0\rho_a v_0^2 l^2 ab\begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & c_0^{-1} - m(1)(4\pi\rho_a ab)^{-1}\end{bmatrix}$$

where α is a positive constant < 1.

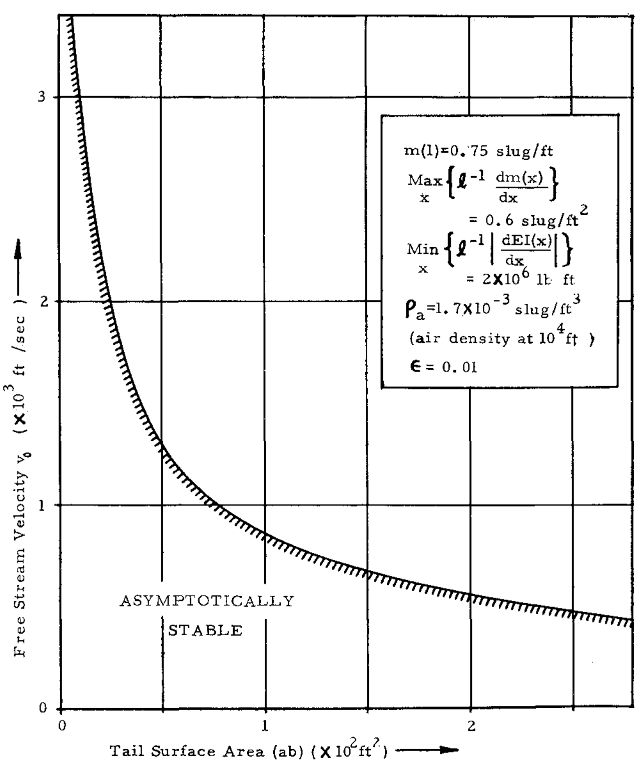

Fig. 2 Stability boundary defined by (15) in the V_0-(ab) parameter plane.

Clearly, dV/dt will be < 0 along any perturbed motion if Q is positive definite and P is positive definite for all $t > t_0$ and all $x\epsilon[0, 1]$, or if the following inequalities are satisfied:

$$c_0 < 4\left[1 + \frac{m(1)}{\rho_a ab}\right]^{-1} \tag{13}$$

$$\left.\begin{aligned} &2(1-\alpha)\left(\min_x\left|\frac{dEI(x)}{dx}\right|\right)\left[lk_d(t,x) + \frac{1}{2}c_0v_0\frac{dm(x)}{dx}\right] > c_0v_0l^4k_d{}^2(t,x) \\ &(1-\alpha)\left[2c_0\alpha\left(lk_d(t,x) + \frac{1}{2}v_0\frac{dm(x)}{dx}\right)\left(\min_x\left|\frac{dEI(x)}{dx}\right|\right) - v_0{}^3(\pi\rho_a lab)^2\right] - \alpha v_0c_0{}^2l^4k_d{}^2(t,x) > 0 \end{aligned}\right\} \begin{aligned}&\text{for all } t > t_0\\ &\text{all } x\epsilon[0,1]\end{aligned} \tag{14}$$

The preceding inequalities are deduced from those of Sylvester for positive definiteness of symmetric matrices. By choosing a constant c_0 satisfying both (7) and (13), the inequalities (11) and (14) become a sufficient condition for asymptotic stability. Note that since $(1 - \alpha)v_0{}^3(\pi\rho_a lab)^2 > 0$, only the second inequality in (14) needs to be considered. For the special case where the distributed damping coefficient $k_d \equiv 0$, α and c_0 may be taken to be

$$\alpha = \tfrac{1}{2} \qquad c_0 = (4 - \epsilon)\{1 + [m(1)/\rho_a ab]\}^{-1}$$

where ϵ is any positive number <4. Thus, condition (14) reduces to

$$v_0 < (2\pi l)^{-1}\left[(4-\epsilon)\rho_a{}^{-1}\frac{dm(x)}{dx}\left(\min_x\left|\frac{dEI(x)}{dx}\right|\right)\right]^{1/2} \times [ab(\rho_a ab + m(1))]^{-1/2} \tag{15}$$

The region of asymptotic stability in the v_0 - (ab) parameter plane represented by (15) with a typical set of parameters is shown in Fig. 2. The result agrees with physical reasoning that the stability boundary for v_0 should tend to $+\infty$ as the tail surface area $(ab) \rightarrow 0$. Since the exact stability boundary for this system is not known,‡ nothing can be said about the sharpness of condition (15).

Finally, it should be remarked that the selection of the functional V was by no means a straightforward task. In fact, among the class of quadratic functionals of the state variables, only (5) was found to be useful. Unfortunately, there are no systematic methods for constructing the required functionals starting from the system equations. This, of course, is the main difficulty in applying Lyapunov's direct method. On the other hand, sufficient conditions for asymptotic stability of equilibrium of more complex flexible vehicles have been obtained using the approach outlined here.[4] Since this method is valid for a rather wide class of dynamical systems, further investigations on its applications to aeroelastic stability problems may be fruitful.

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‡ Note that a stability analysis based on an approximate finite-dimensional mathematical model of (1–3) generally leads to conditions that are insufficient for asymptotic stability.

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A Single Formula for the Velocity Distribution in the Turbulent Inner and Outer Boundary Layers

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Introduction

In applying the integral method for determining turbulent boundary-layer gross properties, it is necessary to know the velocity distribution in the viscous sublayer and turbulent inner and outer layers. More than a dozen investigators have proposed various analytical forms for the velocity profile, but none of them offer a simple single formula that can be used directly in the integral method. The nature of these various profiles is reviewed and summarized in the survey articles by Kestin and Richardson,[1] and Spalding and Chi.[2] It must be noted that all the investigators, except Reichardt,[3] van Driest,[4] and Spalding,[5] introduced two or more separate expressions in order to describe the velocity distribution in the entire three regions.

For the convenience of integration across the boundary layer, it is desirable to express the velocity profile by one simple mathematical formula rather than by two or more discontinuous formulas. However, both van Driest's and Spalding's formulas are impractical for the direct application of the integral method because the former involves the derivative dy_*/du_*, and the latter gives y_* as a function of u_* instead of vice versa. Moreover, Reichardt's and Spalding's velocity profiles, as shown in Figs. 1 and 2, do not satisfy the defect law in the outer portion of the turbulent boundary layer. Since the major portion of the momentum in the boundary layer is contained in this turbulent outer layer, Reichardt's and Spalding's formulas are not satisfactory for the solution of momentum integral equation.

A more suitable velocity profile in the form that can be used in the integral method is recently derived by Libby et al.[6] Their result does describe both "the law of the wall" and "the defect law" adequately, but, three different forms including a complicated expression for the outer layer are required to represent the velocity profile completely.

Instead of using an analytical expression, Coles[7] has succeeded in describing the complete velocity profile with a numerically tabulated wake function. To achieve the analytical integration of the momentum equation across the boundary-layer thickness, some modifications of Coles' approach are attempted in the present study.

Formulation and Derivation

Based upon the mathematical and physical arguments, the new expression should be smooth and continuous over the region of the boundary layer with an analytical form permitting easy integration across the boundary-layer thickness. It should satisfy the conditions that 1) the normalized

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